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Hausdorff Dimensions in Two-Dimensional Maps and Thermodynamic Formalism

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We compute numerically the Hausdorff dimensions of the Gibbs measures on the invariant sets of Axiom A systems. In particular, we stress the existence of a measure which has maximal dimension and can be relevant for the ergodic properties of the system. For hyperbolic maps of the plane with constant Jacobian j, we apply the Bowen–Ruelle formula, using the relation $F(\beta = d_H - 1) = \ln j$, which links the Hausdorff dimension d_H of an attractor to a free energy functional $F(\beta)$ defined in the thermodynamic formalism. We provide numerical evidence that this relation remains valid for some nonhyperbolic maps, such as the Hénon map.

KEY WORDS: Strange attractors; thermodynamic formalism; Gibbs states; Hausdorff dimension; generalized Lyapunov exponents.

1. INTRODUCTION

The Bowen–Ruelle relation[1] is one of the most important tools for the analysis of the fractal properties of mixing repellors such as Julia sets of polynomial maps. In fact, it gives the Hausdorff dimension d_H of invariant sets in terms of a free energy functional defined in the thermodynamic formalism.[2] In this paper, we want to show how to apply it to more general Axiom A systems. Roughly speaking, Axiom A means that the invariant set S of the map F is hyperbolic, i.e., at each point x of S, the tangent space is the direct sum of two subspaces, varying continuously with x, the stable subspace where initial vectors are exponentially contracted

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and the unstable one, where they are exponentially stretched. Moreover, the periodic points of F are dense in S.

In the following we focus our discussion on the attractors of two-dimensional maps, where our result is a simple but important achievement both for analytical calculations and for numerical purposes.

We in fact indicate how to implement the Bowen–Ruelle formula via the generalized Lyapunov exponents $L(q)$,[3] and show that its correction to the Kaplan and Yorke formula[4] (giving the information dimension d_I, i.e., the Hausdorff dimension of the smallest set of full "physical" measure) can be found in terms of the variance of the finite-time fluctuations of the Lyapunov exponent.

Moreover, we find a simple method to give a thermodynamical description of the ergodic properties of two-dimensional maps by means of the Hausdorff dimensions of a family of ergodic invariant probability measures, the equilibrium Gibbs states.[6] This permits us to derive some known results of ergodic theory in a simple way and to provide numerical evidence of the existence of an important measure, the Gibbs state with maximum Hausdorff dimension. We also try to calculate those Hausdorff dimensionalities for the Hénon map, a polynomial nonhyperbolic map of the plane. In this case we are able to give a rough characterization of the nonhyperbolicity.

2. EQUILIBRIUM STATES AND THEIR DIMENSIONALITIES

Let us recall that the equilibrium Gibbs state of the function ϕ is the F-invariant ergodic measurement on S which realizes the supremum in the variational principle for the topological pressure[2]:

$$P(F, \phi) = \sup_{\mu \in M_E(S)} \left[K(\mu) + \int d\mu(x)\, \phi(x) \right] \tag{2.1}$$

in the set $M_E(S)$ of the F-invariant measures on S. Here $K(\mu)$ is the Kolmogorov–Sinai entropy of μ.[13]

In the following we shall consider the interaction function $\phi(x) = -\beta \ln \|D_x F | E_x^U\|$ (where $\|D_x F | E_x^U\|$ is the norm of the tangent map of the transformation F restricted to the unstable subspace E_x^U) and the corresponding equilibrium Gibbs states μ_β. Note the naive thermodynamic analogy: β is an inverse temperature, $F(\beta) = -P(\beta)/\beta$ is the free energy, with $P(\beta) = P(F, \phi = -\beta \ln \|D_x F | E_x^U\|)$, while $K(\mu_\beta)$ is the entropy, and the maximal Lyapunov exponent

$$\lambda^+(\mu_\beta) = \int d\mu_\beta(x) \ln \|D_x F | E_x^U\| \tag{2.2}$$

is the internal energy.

In the literature there are two different ways of computing the free energy by numerical experiments:

1. A thermodynamic approach via the computation of the free energy. The pressure is derived from the generalized Lyapunov exponents $L(q)$ defined as[3]

$$L(q)=\lim_{t\to\infty}\frac{1}{t}\ln\int d\rho(x)\,\|D_xF^t\,|\,E_x^{\mathrm{U}}\|^q \tag{2.3}$$

where ρ is the physical measure singled out by the time averages in real numerical experiments.

$L(q)$ is a convex function of q and in the limit $q\to 0$ one has

$$\lambda^+(\rho)=\lim_{q\to 0}\frac{dL}{dq}=\lim_{q\to 0}\frac{L(q)}{q}$$

In the invariant sets of Axiom A systems, the $L(q)$ are related to the free energy via the relation[8,9]

$$L(q)=P(1-q) \tag{2.4}$$

By using (2.4) and the variational principle (2.1), one can obtain the maximal Lyapunov exponent of μ_β as

$$\lambda^+(\mu_\beta)=-\frac{dP}{d\beta}=\left.\frac{dL}{dq}\right|_{q=1-\beta}$$

and thus the Kolmogorov entropy via the thermodynamic relation $K(\mu_\beta)=P(\beta)+\beta\lambda^+(\mu_\beta)$.

2. A statistical approach via the computation of the Gibbs states. Axiom A attractors are given by the closure of the fixed points of F^n for $n\geqslant 0$.[2] It is therefore possible to approximate the Gibbs states by weighted delta function concentrated on the points z_i such that $z_i=F^nz_i$,

$$d\mu_\beta(x)=\lim_{n\to\infty}Z_n(\beta)^{-1}\sum_{z_i}\delta(z_i-x)\exp(-\beta\ln\|D_{z_i}F^n\,|\,E_{z_i}^{\mathrm{U}}\|)\,dx \tag{2.5}$$

where

$$Z_n(\beta)=\sum_{z_i}\exp(-\beta\ln\|D_{z_i}F^n\,|\,E_{z_i}^{\mathrm{U}}\|) \tag{2.6}$$

and

$$P(\beta)=\lim_{n\to\infty}\frac{1}{n}\ln Z_n(\beta)$$

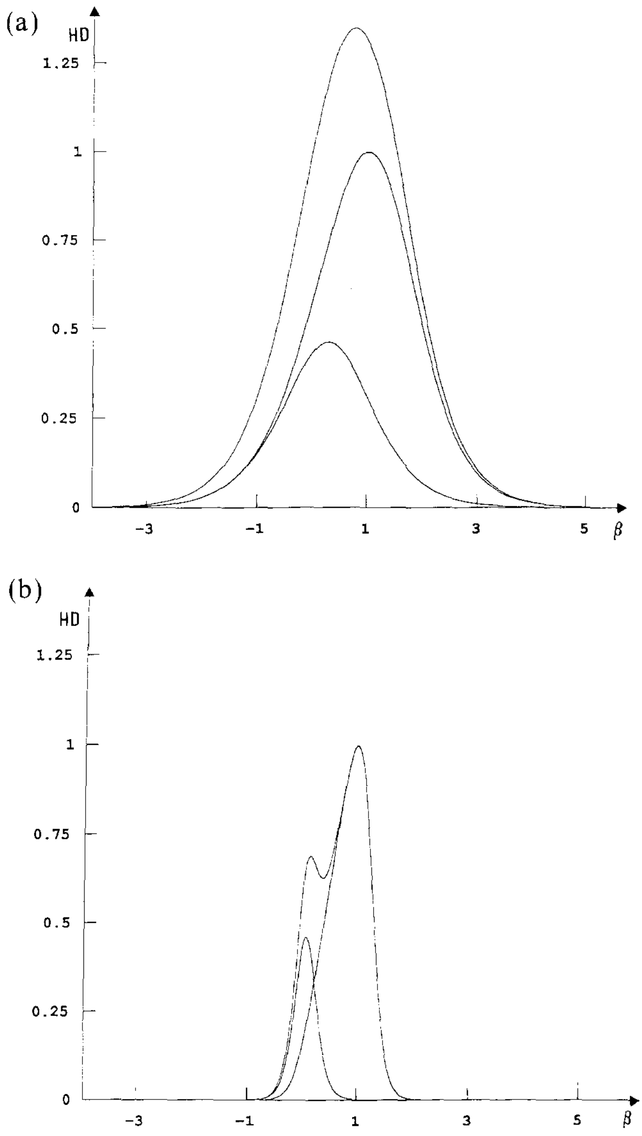

Fig. 1. (a) HD(β) versus β for the baker transformation with $\alpha = 0.3$, $\gamma_a = 0.1$, and $\gamma_b = 0.4$. The tangent map has nonconstant determinant. The three lines are HD, $\mathrm{HD^U}$, and $\mathrm{HD^{ST}}$. The maximum of $\mathrm{HD^{ST}}$ is $d_{\mathrm{H}}^{\mathrm{ST}} = d_{\mathrm{H}} - 1 = 0.46$, solution of the equation $\gamma_a^{d_{\mathrm{H}}^{\mathrm{ST}}} + \gamma_b^{d_{\mathrm{H}}^{\mathrm{ST}}} = 1$. One sees that the maximum of $\mathrm{HD^U}$ is equal to 1 and is reached for $\beta = 1$, as in all the following cases. (b) As for (a), but with $\alpha = 10^{-4}$. In this case the functions $\mathrm{HD^{ST}}$ and $\mathrm{HD^U}$ become very peaked around their maxima. The maximum of $\mathrm{HD^{ST}}$ is reached for β, which tends to 0 (maximum entropy Gibbs state) when $\alpha \to 0$. Note that the Hausdorff dimension of the invariant set is the same as in (a), since one has the same γ_a and γ_b values.

This procedure has been recently developed in many cases, and has also been applied to nonhyperbolic systems, where it is nonrigorous, but seems to give good results,[14] assuming that the physical measure ρ is picked up by $\beta = 1$ in (2.5).

The main results of this section is the numerical characterization of the equilibrium states by means of their Hausdorff dimensions or information dimensions, $\mathrm{HD}(\beta) = \inf_J$ (Hausdorff dimension of J), where J are the subsets of S of full measure $\mu_\beta(J) = 1$. For this purpose we can use the Young relation[16] for invariant sets of C^2 diffeomorphisms F of a surface:

$$\mathrm{HD}(\mu) = K(\mu)\left(\frac{1}{\lambda^+(\mu)} - \frac{1}{\lambda^-(\mu)}\right) \tag{2.7}$$

where μ is any F-ergodic measure on A, and $\lambda^+(\mu)$ and $\lambda^-(\mu)$ are, respectively, the largest and smallest Lyapunov exponent. In the case of the physical measure, i.e., $\rho = \mu_{\beta=1}$, (2.7) becomes the Kaplan and Yorke relation[4] for the information dimension:

$$d_I = \mathrm{HD}(\rho) = 1 + \frac{\lambda^+(\rho)}{|\lambda^-(\rho)|} \tag{2.8}$$

We have here used the Pesin equality.[17] which states that the Kolmogorov entropy of ρ is equal to the sum of the positive Lyapunov exponents. For two-dimensional hyperbolic diffeomorphisms it implies $K(\rho) = \lambda^+(\rho)$ and $P(1) = 0$.

We can compute $\mathrm{HD}(\beta)$ as function of β from the $L(q)$ as well as from the fixed point approach. Figure 1 shows $\mathrm{HD}(\beta)$ for the baker's transformation:

$$x_{n+1} = \gamma_a x_n, \qquad y_{n+1} = \frac{1}{\alpha} y_n \qquad \text{if} \quad y_n \leqslant \alpha$$

$$x_{n+1} = \gamma_b x_n, \qquad y_{n+1} = \frac{1}{1-\alpha}(y_n - \alpha) \qquad \text{if} \quad y_n > \alpha$$

where $0 \leqslant x_n, y_n \leqslant 1$, $\gamma_a < \gamma_b < 0.5$, and $\alpha \leqslant 0.5$. In this case we can obtain $\mathrm{HD}(\beta)$ by a direct use of (2.4), since one has a closed form for the topological pressure,[8,9] i.e., $P(\beta) = \ln[\alpha^\beta + (1-\alpha)^\beta]$ (and so for λ_+ and K) and for the negative Lyapunov exponent. In fact, the periodic points of F^n are distributed uniformly in the atoms of the partition $F^{-n}([0, 1] \times [0, 1]) \vee F^n([0, 1] \times [0, 1])$ and a calculation similar to that of the Appendix of ref. 8 gives:

$$\lambda^-(\mu_\beta) = e^{-P(\beta)}(\alpha^\beta \ln \gamma_a + (1-\alpha)^\beta \ln \gamma_b) \tag{2.9}$$

[4] For a critical discussion see ref. 15.

We have also computed HD(β) by means of a numerical algorithm developed by G. Gunaratne that gives the fixed points of F^n together with the tangent map eigenvalues. Figures 2 and 3 show, respectively, the results for the Lozi map with $n=20$ (22,128 fixed points, corresponding to a topological entropy $h_{\mathrm{TOP}} = \ln 22{,}128/\ln 2^{20} = 0.5002$) and for the Hénon map with $n=18$ (4261 fixed points corresponding to $h_{\mathrm{TOP}} = \ln 4261/\ln 2^{18} = 0.4643$). In all the figures one also sees two other curves, the Hausdorff dimensions of the Gibbs states along the stable and unstable directions, i.e., $\mathrm{HD}^{\mathrm{U}}(\mu) = K(\mu)/\lambda^{+}(\mu)$ and $\mathrm{HD}^{\mathrm{ST}}(\mu) = -K(\mu)/\lambda^{-}(\mu)$, as hyperbolic attractors can be regarded as given locally by the product of a continuum (the unstable manifold) times a fractal Cantor set (the intersection of the attractor with the stable manifold).

Let us now stress some common features of the functions shown in Figs. 1–3:

1. The maximum of the HD^{U} is equal to 1 and is reached for $\beta = 1$.
2. The maximum of the $\mathrm{HD}^{\mathrm{ST}}$ is equal to the dimension $d_{\mathrm{H}} - 1$ of the stable Cantorian direction and, if the Jacobian of the transformation is constant, it is reached for $\beta = d_{\mathrm{H}} - 1$.

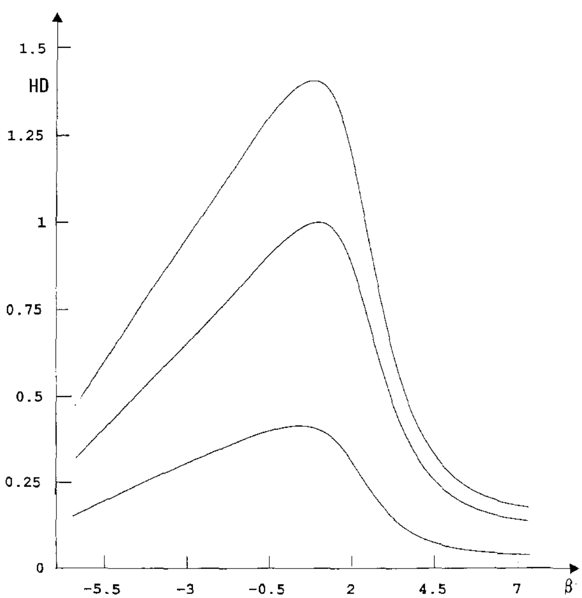

Fig. 2. HD(β) versus for the Lozi map. The three lines are HD, HD^{U}, and $\mathrm{HD}^{\mathrm{ST}}$. In this case as well as for the Hénon map the maximum of $\mathrm{HD}^{\mathrm{ST}} = d_{\mathrm{H}}^{\mathrm{ST}} = 0.4135$ is reached for $\beta = d_{\mathrm{H}}^{\mathrm{ST}}$, since those maps have constant Jacobian.

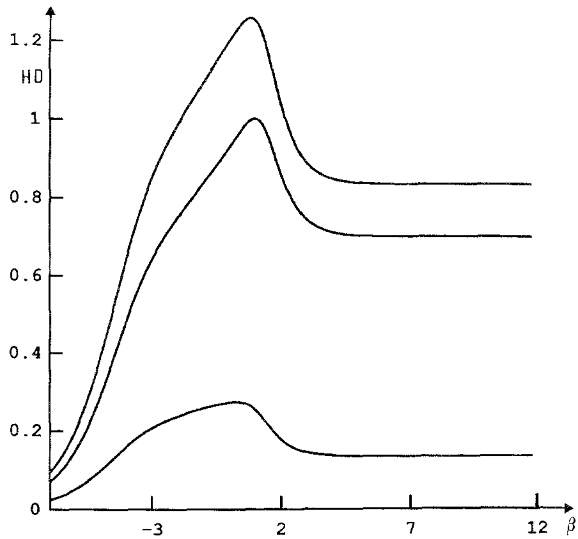

Fig. 3. HD(β) versus β for the Hénon map, as obtained by the Young formula. The three lines are HD, HD^U, and HD^{ST}. The maximum of HD^{ST} is $d_H^{ST} = 0.273$. For $\beta \to \infty$, HD does not vanish and one observes the three asymptotic limits $HD^{ST} = 0.11$, $HD^U = 0.70$, and $HD = 0.81$.

3. There are no equilibrium states for which $HD = d_H$, and the maximum of HD is reached by a measure different from the physical measure.

4. In the Hénon map HD does not vanish when $\beta \to \infty$. We conjecture that this a consequence of the nonhyperbolicity of the map.

The shape of the function HD provides a qualitative characterization of a large class of dynamical systems, since it is a quantity easily accessible to numerical analysis.

Property 1 is just a numerical check that the numerical agorithm works, since the (physical) measure $\mu_{\beta=1}$ is absolutely continuous with respect to the Lebesgue measure along the unstable direction, and so its dimensionality is one. Property 3 provides numerical evidence for the existence of a relevant ergodic measure: the Gibbs state which maximizes the Hausdorff dimension HD(β). Only if the physical measure is uniform also along the stable directions does one have that the maximum of HD is equal to the Hausdorff dimension of the invariant set S. Indeed, it has been shown that generically for F in a C^2 open, dense set[23]

$$\sup_{\mu \in M_E(S)} HD(\mu) < d_H$$

We shall prove at the end of section 3 that HD is maximized for β in the interval $[d_{\mathrm{H}}^{\mathrm{ST}}, 1]$.

Finally, we conjecture that the degree of nonhyperbolicity of a map can be characterized by property 4. The limit of HD(β) for $\beta \to \infty$ is a first indication of the dimensionality of the measure which lives on the ensemble of unstable fixed points with eigenvalue of the tangent map $\to 1$. The larger the limit, the more important the role of nonhyperbolicity. It is interesting to understand if a zero limit value of HD is a necessary condition for the uniqueness of the Gibbs states $\forall \beta$. For instance, Fig. 2 show that the decay of HD(β) toward zero is very slow in the Lozi map, which is quasihyperbolic, and which has an infinite Markov partition(19) corresponding in statistical mechanics language to infinite-range interactions, but probably with such a fast decay that phase transition phenomena cannot be present in the generalized exponents, in contrast with those observed in the Hénon map.(20, 5)

3. A BOWEN–RUELLE FORMULA FOR TWO-DIMENSIONAL ATTRACTORS

In this section we apply the Bowen–Ruelle formula to Axiom A attractors S of hyperbolic C^2 maps of the plane. For these systems, there always exists a unique ergodic measure μ which realizes the maximum in the variational principle (2.1). In the following we shall consider the interaction functions $\phi(x) = -\beta \ln \|D_x F | E_x^{\mathrm{U}}\|$ and $\phi^{\mathrm{ST}}(x) = \beta \ln \|D_x F | E_x^{\mathrm{S}}\|$ (where E_x^{S} is the stable subspace at x) with the corresponding pressure functions $P(\beta)$ and $P^{\mathrm{ST}}(\beta)$, and Gibbs states μ_β and μ_β^{ST}. We refer to them as the unstable and stable Gibbs measures, since we are computing the pressure along the stable and unstable directions at $x \in S$. Let us show that μ_β coincides with μ_β^{ST} if the Jacobian is constant. In this case the sum of the two Lyapunov exponents (λ^+ and λ^-) is equal to the logarithm of the Jacobian. One easily gets

$$\begin{aligned} P(\beta) &= \sup_{\mu} \left[K(\mu) - \beta \int d\mu \ln \|D_x F^n | E_x^{\mathrm{U}}\|) \right] \\ &= K(\mu_\beta) - \beta \cdot \lambda^+(\mu_\beta) \\ &= K(\mu_\beta) + \beta \cdot \lambda^-(\mu_\beta) - \beta \cdot \ln(j) \end{aligned} \tag{3.1}$$

It follows that the "stable" pressure is related to the "unstable" pressure by

$$P(\beta) = P^{\mathrm{ST}}(\beta) - \beta \ln j \tag{3.2}$$

[5] See also ref. 21 for a discussion of nonhyperbolic attractors.

since

$$P^{\mathrm{ST}}(\beta) = P(F; \beta \ln \|D_x F | E_x^{\mathrm{S}}\|) = K(\mu_\beta^{\mathrm{ST}}) + \beta \cdot \lambda^-(\mu_\beta^{\mathrm{ST}}) \tag{3.3}$$

One also has from the uniqueness of the Gibbs states

$$\mu_\beta = \mu_\beta^{\mathrm{ST}} \qquad \forall \beta \in \Re \tag{3.4}$$

Let us stress that this is not the case if the Jacobian j is not constant, as, e.g., in the baker transformation, where one sees that $P(\beta) = \ln[\alpha^\beta + (1-\alpha)^\beta]$ and $P^{\mathrm{ST}}(\beta) = -\ln(\gamma_a^\beta + \gamma_b^\beta)$.

We can now use some known results of ergodic theory. In particular, McCluskey and Manning[23] and Pesin[17] have extended the Bowen–Ruelle formula for repellors to the pressure computed along the stable directions, obtaining

$$P(\beta = 1) = 0, \qquad P^{\mathrm{ST}}(\beta = d_{\mathrm{H}}^{\mathrm{ST}}) = 0 \tag{3.5}$$

where $d_{\mathrm{H}}^{\mathrm{ST}} = d_{\mathrm{H}} - 1$ is the Hausdorff dimension of the stable direction of the attractor. As consequence of our previous arguments, we have

$$F(\beta = d_{\mathrm{H}}^{\mathrm{ST}}) = \ln j \tag{3.6a}$$

i.e.,

$$L(1 - d_{\mathrm{H}}^{\mathrm{ST}}) = -d_{\mathrm{H}}^{\mathrm{ST}} \ln j \tag{3.6b}$$

where we have used (2.4), which links the free energy $F(\beta) = -P(\beta)/\beta$ to the generalized Lyapunov exponents $L(q)$. Note also that (3.5) implies $P^{\mathrm{ST}}(1) = -\ln j$, which simply means that the contraction rate of a uniform square of points around the attractors is given by the logarithm of the Jacobian.

The temporal intermittency[3] of the chaotic degree in some sense makes the information dimension different from the Hausdorff dimension of the attractor. In fact, if $L(q) = \lambda q$, i.e., in the absence of temporal intermittency, the free energy would be linear in the temperature β^{-1} and the physical measure would become "uniform" along both unstable and stable manifolds, so that $d_{\mathrm{H}} = \mathrm{HD}(\mu_{\beta=1})$. This, however, is a very exceptional case. In general, we can expand $L(q)$ for small q as

$$L(q) = \lambda q + \tfrac{1}{2}\sigma^2 q^2 + O(q^3) \tag{3.7}$$

and thus compute the correction to the Kaplan and Yorke formula for the information dimension $d_I = -\lambda^+/\lambda^-$ of the stable manifold in terms of σ^2, where σ^2/t is the variance of the finite-time t fluctuations of the Lyapunov exponent.[3] From (3.6b) we obtain the following approximation up to the order $O(1-d_{\rm H}^{\rm ST})^3$:

$$d_{\rm H}^{\rm ST}=d_I+\frac{\sigma^2}{\lambda^+}(d_I+2d_I^2) \tag{3.8}$$

We have checked that for the Lozi map $x_{n+1}=a\,|x_n|+y_n+1$, $y_{n+1}=bx_n$, with $a=1.7$, $b=0.5$:

$$\lambda=0.4702\pm0.0001, \qquad \sigma^2=0.068\pm0.001$$

$$d_I=1.404, \qquad d_{\rm H}=1.413, \qquad \text{approximation (3.8)}=1.410$$

Moreover, the approximation seems to hold also for the Hénon map $x_{n+1}=ax_n^2+y_n+1$, $y_{n+1}=bx_n$, which is not an Axiom A system, where we have found for $a=1.4$, $b=0.3$:

$$\lambda=0.4192\pm0.0002, \qquad \sigma^2=0.136\pm0.001$$

$$d_I=1.255, \qquad d_{\rm H}=1.273, \qquad \text{approximation (3.8)}=1.278$$

We finally want to prove that the β-value which maximizes HD is in the interval $[d_{\rm H}^{\rm ST}, 1]$. Let us in fact recall that the pressure $P(\beta)$ and $P^{\rm ST}(\beta)$ are analytic convex non-increasing functions of β, vanishing respectively for $\beta=1$ and for $\beta=d_{\rm H}^{\rm ST}$[2,23]. Indicating the derivative with respect to β with ∂_β we thus obtain:

$$\partial_\beta \mathrm{HD}(\beta)=\frac{P(\beta)\,\partial_\beta^2 P(\beta)}{(\partial_\beta P(\beta))^2}+\frac{P^{\rm ST}(\beta)\,\partial_\beta^2 P(\beta)}{(\partial_\beta P^{\rm ST}(\beta))^2}$$

This implies that $\partial_\beta\mathrm{HD}$ is strictly positive when $\beta\leqslant d_{\rm H}^{\rm ST}$, and that $\partial_\beta\mathrm{HD}$ is strictly negative, when $\beta\geqslant 1$. For the continuity of $\partial_\beta\mathrm{HD}$, we can conclude that the value β^* maximizing $\mathrm{HD}(\beta)$ should belong to $[d_{\rm H}^{\rm ST}, 1]$.

4. CONCLUSIONS

We have illustrated the theory of Young and of McCluskey and Manning on the information dimension, applying it to low-dimensional strange sets. We have described two practical methods based on the thermodynamic formalism to compute the information dimension $\mathrm{HD}(\mu_\beta)$ of a continuous set of ergodic measures, the Gibbs states μ_β related to an interaction function $\phi(x)=-\beta\ln|D_xF|$. We have provided numerical evidence that the maximum of HD does exist for hyperbolic maps with constant Jacobian and for the baker's transformation as well as for the nonhyperbolic Hénon map. In all those cases, the Gibbs state maximizing $\mathrm{HD}(\beta)$ differs from the physical measure $\mu_{\beta=1}$, which maximizes only $\mathrm{HD}^{\rm U}(\beta)$.

The main result of this paper is the application of the Bowen–Ruelle formula to compute fractal dimensions of hyperbolic attractors. It makes it possible to improve the Kaplan and Yorke formula in two-dimensional mappings, by taking into account the variance of the finite-time fluctuations of the Lyapunov exponent besides its average value.

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